On cotangent manifolds, complex structures and generalized geometry

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Abstract: Given a linear connection D on a manifold M and a smooth field of endomorphisms $\mathcal{J}$ of the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$, such that $\mathcal{J}^2 = -\mathrm{Id}$ and $\mathcal{J}$ is symmetric or skew-symmetric with respect to the canonical metric of neutral signature of $\mathbb{T}M$, we construct an almost complex structure on the cotangent manifold T^*M and we study its integrability. Our approach unifies and generalizes various results already existing in the literature.

1 Introduction

The starting point of this note is a result proved in [1], which states that the cotangent manifold of a special symplectic manifold (M, J, ∇, ω) inherits, under the assumption that $\omega^{1,1}$ is non-degenerate and ∇-parallel, a canonical hyper-Kähler structure (J_1, J_2, g). Recall that a manifold M with a complex structure J, a flat connection ∇ and a symplectic form ω is special symplectic if $d^{\nabla} J = 0$ (i.e. $\nabla_X(J)(Y) = \nabla_Y(J)(X)$, for any $X, Y \in TM$) and $\nabla\omega = 0$. The connection ∇, acting on the cotangent bundle $\pi : T^*M \to M$, induces a decomposition

$$T(T^*M) = H^{\nabla} \oplus \pi^*T^*M = \pi^*(TM \oplus T^*M) \tag{1}$$

into horizontal and vertical subbundles. By means of this decomposition, the hyper-Kähler structure on T^*M is given by (the pull-back of)

$$J_1 := \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix}, \quad J_2 := \begin{pmatrix} 0 & -(\omega^{1,1})^{-1} \\ \omega^{1,1} & 0 \end{pmatrix}, \quad g := \begin{pmatrix} g^{1,1} & 0 \\ 0 & (g^{1,1})^{-1} \end{pmatrix},$$

where $g^{1,1} := \omega^{1,1}(J\cdot, \cdot)$. A key fact in the proof that (J_1, J_2, g) is hyper-Kähler is the integrability of J_1 and J_2. The integrability of J_2 follows from

a local argument, which uses ∇-flat coordinates and $\nabla\omega^{1,1} = 0$. For the integrability of J_1, one notices, using the special complex condition $d^\nabla J = 0$, that $H^\nabla \subset T(T^*M)$ is invariant with respect to the canonical complex structure J_{can} of T^*M induced by J. Hence, J_1 coincides with J_{can} and is integrable. These arguments can be found in [1].

With special geometry as a motivation, in this note we consider the following setting: a manifold M with a linear connection D and a smooth field of endomorphisms of the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$, such that $\mathcal{J}^2 = -\mathrm{Id}$. Following [9] (rather than the usual terminology from generalized geometry), we call $\mathcal{J}$ a generalized almost complex structure. Motivated by J_1 and J_2 above, we assume that $\mathcal{J}$ is symmetric or skew-symmetric with respect to the canonical metric of neutral signature of $\mathbb{T}M$. From D and $\mathcal{J}$ we construct an almost complex structure $J^{\mathcal{J},D}$ on the cotangent manifold T^*M and we study its integrability. This provides a new insight, from the generalized complex geometry point of view, on the above arguments from [1]. It turns out that other results from the literature fit into this framework.

The structure of this note is the following. In Section 2 we prove basic facts we need from generalized geometry. While skew-symmetric generalized complex structures are well-known (see e.g. Gualtieri's thesis [6] for basic facts), the symmetric ones do not seem to appear in the literature. We begin by studying symmetric generalized complex structures on (real) vector spaces. We find the general form of their holomorphic space (see Proposition 4) and we show that any symmetric generalized complex structure on a vector space is, modulo a B-field transformation, the direct sum of one determined by a complex structure and another determined by a pseudo-Euclidian metric (see Examples 6 and Theorem 7). Therefore, there is an obvious analogy with the theory of skew-symmetric generalized complex structures developed in [6] and we discuss it in Subsection 2.2. For our purposes it is particularly relevant the common description of the holomorphic space $L^\tau(E,\epsilon)$ of a symmetric or, respectively, skew-symmetric generalized complex structure on a vector space V, in terms of a complex subspace $E \subset V^{\mathbb{C}}$, with $E + \bar{E} = V^{\mathbb{C}}$, and a skew-Hermitian, respectively skew-symmetric form ϵ on E, satisfying a non-degeneracy condition (see Corollary 8). These results extend pointwise to manifolds (see Subsection 2.3). Despite the above analogies, there is an important difference between symmetric and skew-symmetric generalized almost complex structures: unlike the skew-symmetric ones, the symmetric generalized almost complex structures are never (Courant) integrable.

In Section 3 we prove our main result. Here we determine necessary and sufficient conditions for the almost complex structure $J^{\mathcal{J},D}$ on T^*M, determined by a symmetric or skew-symmetric generalized almost complex

structure $\mathcal{J}$ on a manifold M and a linear connection D on M, to be integrable. We obtain obstructions in terms of the curvature of D and the data (E, ϵ) defining the holomorphic bundle L of $\mathcal{J}$. In particular, the complex subbundle $E \subset T^{\mathbb{C}}M$ must be involutive and ϵ must satisfy a differential equation involving D (see Theorem 14).

Various particular cases of our main result are discussed in Section 4. Besides the above examples coming from special geometry, we consider the case when $M = G$ is a Lie group and $\mathcal{J}$ a (left)-invariant skew-symmetric generalized almost complex structure on G. With the connection D suitably chosen, the integrability of $J^{\mathcal{J},D}$ becomes precisely the Courant integrability of $\mathcal{J}$ (see Proposition 21). We recover the well-known bijective correspondence between invariant skew-symmetric generalized complex structures on G and invariant complex structures on the cotangent group T^*G, skew-symmetric with respect to the natural bi-invariant metric of neutral signature of T^*G (see [2, 3, 8]). Other examples, coming from Hermitian geometry, are also discussed (see Corollary 18).

2 Symmetric generalized complex structures

In this section we develop the basic properties of symmetric generalized complex structures. Subsections 2.1 and 2.2 refer to their algebraic properties, while in Subsection 2.3 we discuss the integrability.

2.1 Linear symmetric generalized complex structures

Let V be a real vector space of dimension $2n$. We denote by

$$g_{\mathrm{can}}(X+\xi, Y+\eta) = \frac{1}{2}\left(\xi(Y)+\eta(X)\right), \quad X+\xi, Y+\eta \in V \oplus V^* \tag{2}$$

the canonical pseudo-metric of signature $(2n, 2n)$ on $V \oplus V^*$.

Definition 1. *A (symmetric, respectively skew-symmetric) generalized complex structure on V is an endomorphism $\mathcal{J} \in \mathrm{End}(V \oplus V^*)$ such that $\mathcal{J}^2 = -\mathrm{Id}$ (symmetric, respectively skew-symmetric with respect to g_{can}).*

Remark 2. In the classical terminology of generalized geometry, a generalized complex structure is automatically assumed to be skew-symmetric. In this note we prefer the language of [9], where generalized complex structures are not assumed, apriori, to be compatible in any way with g_{can}.

In the following proposition we describe the holomorphic space of symmetric generalized complex structures. Before we need to introduce a notation which will be used along the paper.

Notations 3. For a complex subspace $E \subset V^{\mathbb{C}}$, we denote by $\bar{E}$ the image of E through the anti-linear conjugation $V^{\mathbb{C}} \ni X \to \bar{X} \in V^{\mathbb{C}}$ with respect to V. It is a complex subspace of $V^{\mathbb{C}}$.

Proposition 4. *A (complex) subspace L of $(V \oplus V^*)^{\mathbb{C}}$ is the holomorphic space of a symmetric generalized complex structure on V if and only if it is of the form*

$$L = L^{-}(E, \epsilon) := \{X + \xi \in E \oplus (V^{\mathbb{C}})^*, \quad \xi|_{\bar{E}} = i_X \epsilon\}, \tag{3}$$

where E is any complex subspace of $V^{\mathbb{C}}$, such that $E + \bar{E} = V^{\mathbb{C}}$, and $\epsilon \in E^* \otimes \bar{E}^*$ is any complex-bilinear form satisfying the following two conditions:

i) it is skew-Hermitian, i.e.

$$\epsilon(X, \bar{Y}) + \overline{\epsilon(Y, \bar{X})} = 0, \quad \forall X, Y \in E. \tag{4}$$

ii) $\mathrm{Im}(\epsilon|_{\Delta})$ *is non-degenerate. Here $\Delta \subset V$ is the real part of $E \cap \bar{E}$, i.e. $\Delta^{\mathbb{C}} = E \cap \bar{E}$.*

Proof. Let $\mathcal{J}$ be a symmetric generalized complex structure on V, with holomorphic space $L \subset (V \oplus V^*)^{\mathbb{C}}$. We denote by

$$\pi_1 : (V \oplus V^*)^{\mathbb{C}} \to V^{\mathbb{C}}, \quad \pi_2 : (V \oplus V^*)^{\mathbb{C}} \to (V^{\mathbb{C}})^*$$

the natural projections. Define $E := \pi_1(L)$ and let

$$\epsilon : E \to \bar{E}^*, \quad \epsilon(X) := \pi_2 \circ (\pi_1|_L)^{-1}(X)|_{\bar{E}}. \tag{5}$$

We claim that $\epsilon \in E^* \otimes \bar{E}^*$ is well defined. To prove this claim, we first remark that L is g_{can}-orthogonal to $\bar{L}$ (because $\mathcal{J}$ is symmetric), or

$$\xi(\bar{Y}) + \bar{\eta}(X) = 0, \quad \forall X + \xi, Y + \eta \in L. \tag{6}$$

Thus, if $X + \xi_1, X + \xi_2 \in (\pi_1|_L)^{-1}(X)$, i.e. $X + \xi_1, X + \xi_2 \in L$, then, from (6), $\xi_1 = \xi_2$ on $\bar{E}$ and we obtain that ϵ is well-defined, as required. From the very definition of ϵ, $L \subset L^{-}(E, \epsilon)$ and, for dimension reasons, we deduce that $L = L^{-}(E, \epsilon)$. Since L is g_{can}-orthogonal to $\bar{L}$, ϵ must be skew-Hermitian. Moreover, $L \oplus \bar{L} = (V \oplus V^*)^{\mathbb{C}}$ implies that $E + \bar{E} = V^{\mathbb{C}}$. We now claim that $L \cap \bar{L} = \{0\}$ implies that $\mathrm{Im}(\epsilon|_{\Delta})$ is non-degenerate. To prove this claim, assume, by absurd, that there is $X \neq 0$ in the kernel of $\mathrm{Im}(\epsilon|_{\Delta})$. Define $\xi \in (V^{\mathbb{C}})^*$ by

$$\xi(Z) = \overline{\epsilon(X, \bar{Z})}, \quad \xi(\bar{Z}) = \epsilon(X, \bar{Z}), \quad \forall Z \in E.$$

Using that $X \in \mathrm{Ker}(\mathrm{Im}(\epsilon|_\Delta))$, one can check that ξ is well-defined and $X+\xi \in L \cap \bar{L}$, which is a contradiction. The claim follows.

Conversely, it may be shown that any subspace $E \subset V^{\mathbb{C}}$, with $E + \bar{E} = V^{\mathbb{C}}$, and skew-Hermitian form $\epsilon \in E^* \otimes \bar{E}^*$ with the non-degeneracy property *ii)*, define, by (3), the holomorphic space of a symmetric generalized complex structure on V. $\square$

Corollary 5. *Let $\mathcal{J}$ be a symmetric generalized complex structure on V, with holomorphic space $L^-(E,\epsilon)$. Then* $\mathrm{Im}(\epsilon)$ *is a pseudo-Euclidian metric and* $\mathrm{Re}(\epsilon)$ *is a 2-form on Δ (the real part of $E \cap \bar{E}$).*

Proof. Straightforward, from (4) and the non-degeneracy of $\mathrm{Im}(\epsilon|_\Delta)$. $\square$

Examples 6. i) A complex structure J on V defines a symmetric generalized complex structure

$$\mathcal{J} := \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix},$$

where $J^*\xi := \xi \circ J$. Its holomorphic space is $L^-(V^{1,0}, 0) = V^{1,0} \oplus \mathrm{Ann}(V^{0,1})$, where $V^{1,0}$ is the holomorphic space of J.

ii) A pseudo-Euclidian metric, seen as a map $g : V \to V^*$, defines a symmetric generalized complex structure

$$\mathcal{J} := \begin{pmatrix} 0 & g^{-1} \\ -g & 0 \end{pmatrix}.$$

Its holomorphic space is $L^-(V^{\mathbb{C}}, ig^{\mathbb{C}})$, where $g^{\mathbb{C}} \in (V^{\mathbb{C}} \otimes V^{\mathbb{C}})^*$ is the complex linear extension of g.

iii) If $\mathcal{J}$ is a symmetric generalized complex structure, then so is its B-field transformation $\exp(B) \cdot \mathcal{J} := \exp(B) \circ \mathcal{J} \circ \exp(-B)$, where $B \in \Lambda^2(V^*)$ and the B-field action is defined by

$$\exp(B) : V \oplus V^* \to V \oplus V^*, \quad X + \xi \to X + i_X B + \xi.$$

If $L^-(E,\epsilon)$ is the holomorphic space of $\mathcal{J}$, then $L^-(E, \epsilon + B^{\mathbb{C}}|_{E \otimes \bar{E}})$ is the holomorphic space of $\exp(B) \cdot \mathcal{J}$, where $B^{\mathbb{C}} \in \Lambda^2(V^{\mathbb{C}})^*$ is the complex linear extension of B.

In following theorem we show that any symmetric generalized complex structure can be (non-canonically) obtained from a complex structure, a pseudo-Euclidian metric and a B-field transformation.

Theorem 7. *Any symmetric generalized complex structure on a vector space V is a B-field transformation of the direct sum of one determined by a complex structure and another determined by a pseudo-Euclidian metric (as in Examples 6).*

Proof. Let $\mathcal{J} \in \mathrm{End}(V \oplus V^*)$ be a symmetric generalized complex structure, with holomorphic space $L = L^-(E, \epsilon)$ (see Proposition 4). Let Δ be the real part of $E \cap \bar{E}$ (i.e. $\Delta \subset V$ and $\Delta^{\mathbb{C}} = E \cap \bar{E}$) and N a complement of Δ in V. Thus

$$V = \Delta \oplus N, \quad E = \Delta^{\mathbb{C}} \oplus (E \cap N^{\mathbb{C}}).$$

We denote by $\mathrm{pr}_{\Delta^{\mathbb{C}}}$ and $\mathrm{pr}_{N^{\mathbb{C}}}$ the projections from $V^{\mathbb{C}}$ to $\Delta^{\mathbb{C}}$ and $N^{\mathbb{C}}$ respectively. We notice that Δ comes with pseudo-Euclidian metric, namely $\mathrm{Im}(\epsilon|_{\Delta})$, and N with a complex structure J^N, with holomorphic space $E \cap N^{\mathbb{C}}$ (and anti-holomorphic space $\bar{E} \cap N^{\mathbb{C}}$). We claim that there is $B \in \Lambda^2(V^*)$ such that

$$(V, \exp(B) \cdot \mathcal{J}) = (\Delta, \mathrm{Im}(\epsilon|_{\Delta})) \oplus (N, J^N), \tag{7}$$

where $(\Delta, \mathrm{Im}(\epsilon|_{\Delta}))$ and (N, J^N) are considered as vector spaces with symmetric generalized complex structures. In order to prove the claim, recall, from Examples 6, that the holomorphic space of $(\Delta, \mathrm{Im}(\epsilon|_{\Delta}))$ is $L^-(\Delta^{\mathbb{C}}, i\mathrm{Im}(\epsilon|_{\Delta})^{\mathbb{C}})$ and the holomorphic space of (N, J^N) is $(E \cap N^{\mathbb{C}}) \oplus \mathrm{Ann}(\bar{E} \cap N^{\mathbb{C}})$. Relation (7) is equivalent to the following statement:

$$X + \xi \in L^-(E, \epsilon + B^{\mathbb{C}}|_{E \otimes \bar{E}})$$

if and only if

$$\mathrm{pr}_{\Delta^{\mathbb{C}}}(X) + \xi|_{\Delta^{\mathbb{C}}} \in L^-(\Delta^{\mathbb{C}}, i\mathrm{Im}(\epsilon|_{\Delta})^{\mathbb{C}}), \quad \mathrm{pr}_{N^{\mathbb{C}}}(X) + \xi|_{N^{\mathbb{C}}} \in (E \cap N^{\mathbb{C}}) \oplus \mathrm{Ann}(\bar{E} \cap N^{\mathbb{C}}).$$

One checks immediately that this, in turn, is equivalent to:

$$(B + \mathrm{Re}(\epsilon))^{\mathbb{C}}|_{\Delta^{\mathbb{C}} \times \Delta^{\mathbb{C}}} = 0, \quad (B^{\mathbb{C}} + \epsilon)|_{(E \cap N^{\mathbb{C}}) \times \Delta^{\mathbb{C}}} = 0, \quad (B^{\mathbb{C}} + \epsilon)|_{E \times (\bar{E} \cap N^{\mathbb{C}})} = 0. \tag{8}$$

Hence, we are looking for a (real) 2-form $B \in \Lambda^2(V^*)$ such that (8) is satisfied. Define a real bilinear form B on $V = \Delta \oplus N$, by

$$B(X, Y) := -\mathrm{Re}(\epsilon)(X, Y), \quad B(Z + \bar{Z}, W + \bar{W}) = -2\mathrm{Re}(\epsilon)(Z, \bar{W})$$

and

$$B(X, Z + \bar{Z}) = -B(Z + \bar{Z}, X) = 2\mathrm{Re}(\epsilon)(Z, X),$$

where $X, Y \in \Delta$ and $Z, W \in E \cap N^{\mathbb{C}}$ (any element from N can be uniquely written as a sum $Z + \bar{Z}$ where $Z \in E \cap N^{\mathbb{C}}$). Using that $\epsilon \in E^* \otimes \bar{E}^*$ is skew-Hermitian, it is easy to check that B is skew-symmetric and its complexification satisfies (8). This concludes our claim.

□

2.2 Analogy with skew-symmetric generalized complex structures

The theory from the previous section is similar to the theory of skew-symmetric generalized complex structures developed by Gualtieri in [6] and allows a unified treatment of these two types of structures. It is well-known that complex and symplectic structures define skew-symmetric generalized complex structures and this corresponds to Example 6 *i)* and *ii)*. In the same framework, Theorem 7 above is analogous to Theorem 4.13 of [6], which states that any skew-symmetric generalized complex structure, is, modulo a B-field transformation, the direct sum of a skew-symmetric generalized complex structure of symplectic type and a one of complex type.

The following unified description of the holomorphic space of symmetric and skew-symmetric generalized complex structures is a rewriting of Proposition 4 from the previous section and of Propositions 2.6 and 4.4 of [6]. We shall use it in our main result (Theorem 14).

Corollary 8. *A complex subspace $L \subset (V \oplus V^*)^{\mathbb{C}}$ is the holomorphic space of a symmetric or, respectively, skew-symmetric generalized complex structure if and only if it is of the form*

$$L = L^{\tau}(E, \epsilon) = \{X + \xi \in E \oplus (V^{\mathbb{C}})^*, \quad \xi|_{\tau(E)} = i_X(\epsilon)\} \tag{9}$$

where $E \subset V^{\mathbb{C}}$ is a complex subspace such that $E + \bar{E} = V^{\mathbb{C}}$ and $\epsilon \in E^* \otimes \tau(E)^*$ is complex bilinear, such that $\mathrm{Im}(\epsilon|_{\Delta})$ *is non-degenerate (where $\Delta \subset V$, $\Delta^{\mathbb{C}} = E \cap \bar{E}$) and*

$$\epsilon(X, \tau(Y)) + \tau(\epsilon(Y, \tau(X))) = 0, \quad \forall X, Y \in E. \tag{10}$$

In (9) and (10) the maps $\tau : V^{\mathbb{C}} \to V^{\mathbb{C}}$ and $\tau : \mathbb{C} \to \mathbb{C}$ are both complex conjugations, respectively both identity maps.

2.3 Remarks on integrability

The generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$ of a smooth manifold M has a canonical metric of neutral signature, defined like in (2), and the theory developed in the previous sections extends pointwise to manifolds, in an obvious way. In particular, the holomorphic bundle L of a symmetric or skew-symmetric generalized almost complex structure on M may be given in terms of a complex subbundle $E \subset T^{\mathbb{C}}M$ (the image of L through the natural projection $(TM \oplus T^*M)^{\mathbb{C}} \to T^{\mathbb{C}}M$) and a section $\epsilon \in \Gamma(E^* \otimes \tau(E)^*)$, satisfying the algberaic properties from Corollary 8 (we assume that all points are

regular, i.e. E is a genuine complex vector bundle). However, in the setting of manifolds we have to consider a new ingredient: the Courant bracket. It is defined on sections of $\mathbb{T}M$ by

$$[X+\xi, Y+\eta] = [X,Y] + L_X\eta - L_Y\xi + \frac{1}{2}\left(\xi(Y) - \eta(X)\right).$$

Definition 9. *A generalized almost complex structure on a manifold M is called integrable (or simply a generalized complex structure) if the space of sections of its holomorphic bundle is closed under the Courant bracket.*

The integrability for skew-symmetric generalized almost complex structures of complex or symplectic type reduces to the usual integrability for almost complex and almost symplectic structures. More generally, the following holds (see Proposition 4.19 of [6]):

Proposition 10. *A skew-symmetric generalized almost complex structure on a manifold M, with holomorphic bundle $L = L(E,\epsilon)$, is integrable, if and only if the subbundle $E \subset T^{\mathbb{C}}M$ is involutive and $d_E\epsilon = 0$, where $d_E\epsilon \in \Lambda^3(E^*)$ is the exterior differential of ϵ along E, defined by*

$$\begin{aligned}(d_E\epsilon)(X,Y,Z) := {} & X\left(\epsilon(Y,Z)\right) + Z\left(\epsilon(X,Y)\right) + Y\left(\epsilon(Z,X)\right)\\ & + \epsilon(X,[Y,Z]) + \epsilon(Z,[X,Y]) + \epsilon(Y,[Z,X])\end{aligned}$$

for any $X,Y,Z \in \Gamma(E)$.

As opposed to Proposition 10, the following holds.

Lemma 11. *Any symmetric generalized almost complex structure is non-integrable.*

Proof. As proved in Proposition 3.26 of [6], a Courant integrable subbundle of $\mathbb{T}^{\mathbb{C}}M$ is either isotropic (with respect to g_{can}) or of the form $(\Delta \oplus T^*M)^{\mathbb{C}}$, where $\Delta \subset TM$ is involutive (and non-trivial). Hence, it cannot be the holomorphic bundle L of a symmetric generalized almost complex structure (recall that L is g_{can}-orthogonal to $\bar{L}$ and $L \oplus \bar{L} = \mathbb{T}^{\mathbb{C}}M$).

□

3 The main result

Let $(M, \mathcal{J}, D)$ be a manifold with a generalized almost complex structure $\mathcal{J}$ and linear connection D. The connection D acts on the cotangent bundle $\pi : T^*M \to M$ and induces a decomposition

$$T(T^*M) = H^D \oplus T^{\mathrm{vert}}(T^*M) = \pi^*(\mathbb{T}M) \tag{11}$$

into horizontal and vertical subbundles. Above, we identified the horizontal bundle H^D with $\pi^*(TM)$. By means of (11), we shall systematically identify, without mentioning explicitly, the tangent bundle of the cotangent manifold T^*M with $\pi^*(\mathbb{T}M)$.

Definition 12. *The almost complex structure $J^{\mathcal{J},D} := \pi^*(\mathcal{J})$ on T^*M is called the almost complex structure defined by $\mathcal{J}$ and D.*

In this section we study the integrability of $J^{\mathcal{J},D}$, under the assumption that $\mathcal{J}$ is symmetric or skew-symmetric. We begin by fixing notations.

Notations 13. In computations, we shall use the notation $\tilde{X}$ for the D-horizontal lift of a vector field $X \in \mathcal{X}(M)$. Sections of $\pi : T^*M \to M$ will be considered as constant vertical vector fields on the cotangent manifold T^*M. With these conventions, the various Lie brackets $[\cdot,\cdot]_{\mathcal{L}}$ of vector fields on T^*M are computed as follows:

$$[\tilde{X},\tilde{Y}]_{\mathcal{L}}(\gamma) = \widetilde{[X,Y]}_{\gamma} - R^D_{X,Y}(\gamma), \quad [\tilde{X},\alpha]_{\mathcal{L}} = D_X(\alpha), \quad [\alpha,\beta]_{\mathcal{L}} = 0 \tag{12}$$

for any $X, Y \in \mathcal{X}(M)$, $\alpha, \beta \in \Omega^1(M)$ and $\gamma \in T^*M$, where

$$R^D_{X,Y} := -D_X D_Y + D_Y D_X + D_{[X,Y]}$$

is the curvature of D.

The main result from this section is the following.

Theorem 14. *Let $(M, \mathcal{J}, D)$ be a manifold with a symmetric or skew-symmetric generalized almost complex structure $\mathcal{J}$ and a linear connection D. Let $L^\tau(E,\epsilon)$ be the holomorphic bundle of $\mathcal{J}$, where $E \subset T^{\mathbb{C}}M$ and $\epsilon \in \Gamma(E^* \otimes \tau(E)^*)$ satisfy the algebraic properties from Corollary 8.*

The almost complex structure $J^{\mathcal{J},D}$ from Definition 12 is integrable, if and only if the following conditions hold:

i) E is an involutive subbundle of $T^{\mathbb{C}}M$;

ii) $R^D|_{E\times E} = 0$ and $D_{\Gamma(E)}\Gamma(\tau(E)) \subset \Gamma(\tau(E))$;

iii) define $\tilde{\epsilon} : E \times E \to \mathbb{C}$ and $\tilde{D} : \Gamma(T^{\mathbb{C}}M) \times \Gamma(T^{\mathbb{C}}M) \to \Gamma(T^{\mathbb{C}}M)$ by

$$\tilde{\epsilon}(X,Y) := \epsilon(X,\tau(Y)), \quad \tilde{D}_X(Y) := \tau D_X(\tau(Y)).$$

Then, for any $X, Y, Z \in \Gamma(E)$,

$$(\tilde{D}_X\tilde{\epsilon})(Y,Z) - (\tilde{D}_Y\tilde{\epsilon})(X,Z) + \tilde{\epsilon}(T^{\tilde{D}}_X Y, Z) = 0. \tag{13}$$

In (13), $\tilde{D}_X\tilde{\epsilon}$ and $T^{\tilde{D}}$ are defined in the usual way:

$$(\tilde{D}_X\tilde{\epsilon})(Y,Z) := X\tilde{\epsilon}(Y,Z) - \tilde{\epsilon}(\tilde{D}_XY,Z) - \tilde{\epsilon}(Y,\tilde{D}_XZ),$$
$$T^{\tilde{D}}_XY := \tilde{D}_XY - \tilde{D}_YX - [X,Y].$$

Proof. We need to prove that $\pi^*L^\tau(E,\epsilon)$ is integrable with respect to the Lie bracket $[\cdot,\cdot]_{\mathcal{L}}$ on vector fields on the cotangent manifold T^*M, if and only if the conditions *i)*, *ii)* and *iii)* hold. Remark, also, that in order to check the integrability of $\pi^*L^\tau(E,\epsilon)$, it is sufficient to consider only basic sections of $\pi^*L^\tau(E,\epsilon)$ (viewed as vector fields on T^*M), i.e. sections of $L^\tau(E,\epsilon)$, lifted to $\pi^*L^\tau(E,\epsilon)$. Therefore, let $X+\xi, Y+\eta \in L^\tau(E,\epsilon)$, i.e. $X,Y\in\Gamma(E)$, $\xi,\eta\in(T^{\mathbb{C}}M)^*$ and

$$\xi(\tau(Z)) = \epsilon(X,\tau(Z)), \quad \eta(\tau(Z)) = \epsilon(Y,\tau(Z)), \quad \forall Z\in\Gamma(E). \tag{14}$$

From (12),

$$[\tilde{X}+\xi,\tilde{Y}+\eta]_{\mathcal{L}}(\gamma) = \widetilde{[X,Y]}_\gamma - R^D_{X,Y}(\gamma) + D_X(\eta) - D_Y(\xi). \tag{15}$$

We obtain that $[X+\xi,Y+\eta]_{\mathcal{L}}$ is a section of $\pi^*L^\tau(E,\epsilon)$ if and only if the right hand side of (15) belongs to the fiber of $L^\tau(E,\epsilon)$ at $\pi(\gamma)$, for any $\gamma\in T^*M$, i.e.

$$[X,Y]\in\Gamma(E), \quad R^D_{X,Y}(\gamma)=0 \tag{16}$$

and

$$(D_X(\eta)-D_Y(\xi))(\tau(Z)) = \epsilon([X,Y],\tau(Z)), \quad \forall Z\in\Gamma(E). \tag{17}$$

We now rewrite (17). From (14), the left hand side of (17) is equal to

$$X\epsilon(Y,\tau(Z)) - Y\epsilon(X,\tau(Z)) - \eta(D_X(\tau(Z))) + \xi(D_Y(\tau(Z)))$$

and (17) becomes

$$X\epsilon(Y,\tau(Z)) - Y\epsilon(X,\tau(Z)) - \eta(D_X(\tau(Z))) + \xi(D_Y(\tau(Z))) = \epsilon([X,Y],\tau(Z)). \tag{18}$$

From (14), $\xi|_{\tau(E)} = i_X\epsilon$, but ξ can take any values on a complement of $\tau(E)$ in $T^{\mathbb{C}}M$. Similarly, the only obstruction on η is its restriction to $\tau(E)$. Using also (16), it follows that $\pi^*L^\tau(E,\epsilon)$ is $[\cdot,\cdot]_{\mathcal{L}}$-involutive if and only if

$$[\Gamma(E),\Gamma(E)]\subset\Gamma(E), \quad R^D|_{E\times E}=0, \quad D_{\Gamma(E)}\Gamma(\tau(E))\subset\Gamma(\tau(E)) \tag{19}$$

and relation (18) holds. The third equality (19) implies that (18) is equivalent to (13) (easy check) and our claim follows. $\square$

Remark 15. In the setting of the above theorem, if $\mathcal{J}$ is skew-symmetric, $\tilde{\epsilon} = \epsilon \in \Gamma(\Lambda^2(E^*))$, $\tilde{D} = D$ (because $\tau : T^{\mathbb{C}}M \to T^{\mathbb{C}}M$ is the identity map) and equation (13) becomes

$$(d\epsilon)(X, Y, Z) - (D_Z\epsilon)(X, Y) - \epsilon(T^D_Z X, Y) - \epsilon(X, T^D_Z Y) = 0, \quad \forall X, Y, Z \in \Gamma(E), \tag{20}$$

where T^D is the torsion of D. In particular, if E is involutive and the connection D is chosen such that

$$R^D|_{E\times E} = 0, \quad D_{\Gamma(E)}\Gamma(E) \subset \Gamma(E) \tag{21}$$

and

$$(D_Z\epsilon)(X, Y) = \epsilon(T^D_X Z, Y) + \epsilon(X, T^D_Y Z), \quad \forall X, Y, Z \in \Gamma(E), \tag{22}$$

then $\mathcal{J}$ is (Courant) integrable if and only if $J^{\mathcal{J},D}$ is integrable (because (20) becomes $d_E\epsilon = 0$, see Proposition 10). In the following section we will give an example when (21) and (22) hold, for a suitable connection D (see Lemma 20).

Remark 16. The Lie bracket $[\cdot,\cdot]_{\mathcal{L}}$ is closely related to a bracket $[\cdot,\cdot]_D$ on sections of $\mathbb{T}M$, defined and studied in [9]:

$$[X + \xi, Y + \eta]_D = [X, Y] + D_X(\eta) - D_Y(\xi). \tag{23}$$

More precisely,

$$[\tilde{X} + \xi, \tilde{Y} + \eta]_{\mathcal{L}}(\gamma) = (\pi^*[X + \xi, Y + \eta]_D)(\gamma) - R^D_{X,Y}(\gamma), \quad \forall \gamma \in T^*M. \tag{24}$$

When D is flat, the $[\cdot,\cdot]_{\mathcal{L}}$-integrability for $J^{\mathcal{J},D}$ becomes the $[\cdot,\cdot]_D$-integrability for $\mathcal{J}$, which was studied in [9] for various classes of $\mathcal{J}$ and under the additional assumption that D is torsion-free. However, we do not assume the flatness (or torsion-free property) of D and we will consider in Section 4 particular cases when $J^{\mathcal{J},D}$ is integrable but D is not flat. In our approach we use as a main tool the simple form of the holomorphic bundle of generalized almost complex structures (symmetric and skew-symmetric), which does not seem to be exploited in [9].

4 Applications

In this section we apply Theorem 14 to various particular cases.

Proposition 17. *Let (M, J, D) be a manifold with an almost complex structure J and linear connection D. The almost complex structure $J^{\pm}$ on T^*M, defined by D and the generalized almost complex structure*

$$\mathcal{J}^{\pm} := \begin{pmatrix} J & 0 \\ 0 & \pm J^* \end{pmatrix}$$

is integrable if and only if J is a complex structure and

$$R^D_{JX,JY} = R^D_{X,Y}, \quad D_X(J)(Y) = \pm J D_{JX}(J)(Y), \quad \forall X, Y \in TM.$$

Proof. We remark that $\mathcal{J}^+$ is symmetric, with holomorphic bundle $T^{1,0}M \oplus \mathrm{Ann}(T^{0,1}M)$, while $\mathcal{J}^-$ is skew-symmetric, with holomorphic bundle $T^{1,0}M \oplus \mathrm{Ann}(T^{1,0}M)$, and, in the notations of Theorem 14, $\epsilon = 0$. From Theorem 14, if $J^{\pm}$ is integrable, then $T^{1,0}M$ is involutive, i.e J is a complex structure. We remark also that $R^D|_{T^{1,0}M, T^{1,0}M} = 0$ if and only if $R^D_{JX,JY} = R^D_{X,Y}$ and $D_{\Gamma(T^{1,0}M)}\Gamma(T^{1,0}M) \subset \Gamma(T^{1,0}M)$ if and only if $D_X(J)(Y) = -J D_{JX}(J)$, while $D_{\Gamma(T^{1,0}M)}\Gamma(T^{0,1}M) \subset \Gamma(T^{0,1}M)$ if and only if $D_X(J)(Y) = J D_{JX}(J)$, for any $X, Y \in TM$. The claim follows from Theorem 14. □

The first statement of the following corollary was proved in [1] using different methods.

Corollary 18. *Consider the setting of Proposition 17.*

i) If (J, D) is a special complex structure, i.e. D is flat, torsion-free and

$$(d^D J)_{X,Y} := D_X(J)(Y) - D_Y(J)(X) = 0, \quad \forall X, Y \in TM,$$

then J^+ is integrable.

ii) If $D = D^g$ is the Levi-Civita connection of an almost Hermitian structure (g, J), then J^+ is integrable if and only if (g, J) is Kähler and J^- is integrable if and only if J is integrable and the curvature of D^g is J-invariant.

iii) If D is the Chern connection of a Hermitian structure (J, g), then $J^{\pm}$ are integrable.

Proof. The claims follow from Proposition 17. For *i)*, we remark that the special complex condition $d^D J = 0$ implies $D_X(J)(Y) = J D_{JX}(J)$ for any $X, Y \in TM$. For *ii)* we use that $D^g_X(J) = -J D^g_{JX}(J)$, for any $X \in TM$, if and only if J is integrable (see [5] or Proposition 1 of [4]). This proves the statement for J^-. The statement for J^+ follows as well: if J is integrable and $D^g_X(J) = J D^g_{JX}(J)$, then $D^g J = 0$ and (g, J) is Kähler. For *iii)* we use that the Chern connection is Hermitian with curvature of type $(1, 1)$. □

The following lemma is a mild improvement of Lemma 6 of [1].

Lemma 19. *Let (M, ω, D) be a manifold with an almost symplectic structure ω and linear connection D. The almost complex structure on T^*M defined by D and the generalized almost complex structure*

$$\mathcal{J}^{\omega} = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}$$

is integrable if and only if D is flat and, for any $X, Y, Z \in \mathcal{X}(M)$,

$$(d\omega)(X, Y, Z) - (D_Z\omega)(X, Y) - \omega(T^D_Z X, Y) - \omega(T^D_Y Z, X) = 0. \tag{25}$$

Proof. The holomorphic bundle of $\mathcal{J}^{\omega}$ is $L(T^{\mathbb{C}}M, i\omega^{\mathbb{C}})$ and the claim follows from Theorem 14 and Remark 15.

□

We now apply our theory to Lie groups. The key fact in the bijective correspondence between generalized complex structures on Lie groups and complex structures on cotangent groups is the integrability of the two structures involved (for the precise statement of this correspondence, see [2, 3, 8] or the introduction). In Proposition 21 we show how this can be obtained from our approach. Below the superscript "left" means that the objects (vector fields, sections, forms, etc.) are left-invariant. First we need the following simple lemma.

Lemma 20. *Let G be a Lie group and $\epsilon \in \Gamma(\Lambda^2 E^*)^{\mathrm{left}}$ a (left-invariant) 2-form on an involutive, left-invariant subbundle $E \subset T^{\mathbb{C}}G$. Let D be the (flat) connection of G, defined by*

$$D_X(Y) = [X, Y], \quad \forall X, Y \in \mathcal{X}(G)^{\mathrm{left}}. \tag{26}$$

Then $D_{\Gamma(E)}\Gamma(E) \subset \Gamma(E)$ and

$$(D_Z\epsilon)(X, Y) = \epsilon(T^D_X Z, Y) + \epsilon(X, T^D_Y Z), \quad \forall X, Y, Z \in \Gamma(E).$$

Proof. The proof is straightforward. □

Proposition 21. *Let G be a Lie group and D the connection (26). A left-invariant skew-symmetric generalized almost complex structure $\mathcal{J}$ on G is (Courant) integrable if and only if the almost complex structure $J^{\mathcal{J},D}$ on T^*G is integrable.*

Proof. Since $\mathcal{J}$ is left-invariant, so are E and $\epsilon \in \Lambda^2(E^*)$, where $L = L(E, \epsilon)$ is the holomorphic bundle of $\mathcal{J}$. The claim follows from Remark 15 and Lemma 20. □

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